BRIAN SKYRMS

ZENO'S PARADOX OF MEASURE

INTRODUCTION

Zeno of Elea is perhaps best known for his four paradoxes of motion: The Dichotomy, Achilles and the Tortoise, The Arrow, and the Stadium. These are, however, supporting pieces in a grand argument against plurality whose keystone is Zeno's paradox of measure.[1] Adolf Grünbaum has, on several occasions,[2] called attention to the importance of this paradox, and explained how the standard measure-theoretic resolution of it became possible only after Cantor's theory of infinite sets.

The argument in question has emerged from Zeno's fragments only after considerable scholarly reconstruction.[3] It is a *reductio ad absurdum* that an extended thing (for simplicity we can think of a unit line segment) can be thought of as made up of an infinite number of parts. The argument goes roughly as follows:

(I) If the parts had finite magnitude, the whole would have infinite magnitude.

(II) If the parts had no magnitude, then the whole would have no magnitude.

It is set up by a prior argument to the effect that if the thing in question is divisible *ad infinitum*, then it can be partitioned into an infinite number of parts.

In the first part of this paper, I will take a closer look at the logic of Zeno's argument, and of the ways in which various schools attempted to escape its embarrassments. In the second part, I will argue that despite the profound achievements of Cantor and Lebesgue, the fundamental spirit of Zeno's paradox is still capable of mischief.

I. ZENO TO EPICURUS

Zeno on Infinite Divisibility

Zeno argues that if the line segment is divisible *ad infinitum* it can be partitioned into an infinite number of parts. His construction is as follows:

R. S. Cohen and L. Laudan (eds.), Physics, Philosophy and Psychoanalysis, 223–254.

1: Partition the line into two segments by bisecting it.
n: Refine the partition gotten at stage $(n-1)$ by bisecting each member of it.
ω: Take the common refinement of all the partitions gotten at finite stages of the process.

The number of parts at stage ω must be greater than any finite number.

This construction uses something slightly stronger than the stated assumption – it requires not only that each part can be divided but also that it can be divided into two equal parts. The bisection of a line segment with ruler and compass is an elementary construction which was familiar at the time. It was perhaps an important part of the motivation for holding that the line segment is divisible *ad infinitum*. For the stated conclusion bisection is not essential, but it can assume importance if one wants to argue by symmetry that each member of the resulting partition has equal length.

Zeno's construction makes the daring leap from potential to actual infinity. Given that one can, for any finite n move from the $(n-1)$th stage in the construction to the nth stage, Zeno proposes an additional move from the totality of the finite stages to an infinite stage. Aristotle resists this move, charging Zeno with misunderstanding the nature of infinity, but from a modern point of view it is perfectly legitimate.

It is worth looking at this construction from an unabashedly modern view to see what it yields. Let us regard the unit line as a set of points, one for each real number in the closed interval [0, 1]. The bisection of this line partitions it into two point sets: [0, 1/2); [1/2, 1]. (Likewise at each stage of refinement of the partition we throw the midpoints in the right hand set. This decision as to where the midpoints go is arbitrary, but some decision must be made if we are to have a genuine partition.) A sequence of sets, the first a member of the first partition . . . the nth a member of the nth partition, such that for each n the $(n+1)$th set is a subset of the nth set, will be called a *chain*. Zeno's construction comes essentially to this: each intersection of such a chain counts as a part of the line on level ω. The collection of such 'ω-parts' is held to be an infinite partition of the unit line.

This construction gets us *something*. What does it get? Do we get all the points on the unit line? Do we get a partition? First, let us notice that *each point on the unit line is a member of some ω-part*. For any point consider the set containing for each finite n, the member of the nth level partition of which that point is a member. This set is a chain, and the given point is in its intersection. Next we see that *each ω-part contains no more than one point*. Between any two real points, there is some finite distance. Therefore, there is some finite stage n in the construction, such that at that stage the points fall into different elements of the partition. Thus they fall into different ω-parts since for them to fall into the same ω-part they would have to both be members of all elements of the chain of which that ω-part is the intersection. The foregoing two propositions show that the ω-parts do indeed form a *partition* of the unit line; i.e., they are a collection of disjoint sets whose union is the set of points on the line. Are the ω-parts gotten by this construction then exactly the sets containing one point? Not quite. All such sets are ω-parts, but some ω-parts are empty. For example, the intersection of the chain: [0, 1], [0, 1/2), [1/4, 1/2), [3/8, 1/2), [7/16, 1/2), . . . is empty. Any point distinct from the midpoint eventually gets squeezed out, and the point 1/2 is not itself in any member of the chain. So the ω-parts, at least in that version of the construction that I have pursued, consist of the empty set together with the unit set of each point on the real line.

I can't resist quoting here part of a passage from *De generatione et corruptione* where Aristotle is recounting a version of Zeno's argument as the argument which persuaded Democritus of the necessity of indivisible magnitudes:

> Suppose then, that it *is* divided; now, what will be left? Magnitude? No, that cannot be, since there will then be something left not divided, whereas it was *everywhere* divisible. But if there is to be no body or magnitude [left] and yet [this] division is to take place, *then either the whole will be made of points*, and then [parts] of which it is composed will have no size, *or* [that which is left] will be *nothing at all*. [A 2, 316^{a}24–30, quoted in Furley (1967), p. 84. Parenthetical insertions are Furley's, emphasis mine.]

It is tempting to ask – but only in the spirit of speculation – how much of the foregoing analysis of Zeno's construction would have been possible for a first-rate mathematician contemporary with Aristotle provided he accepted that the points on a unit line segment can be associated with the numbers, rational and irrational, from zero to one inclusive, and provided he accepted (perhaps only for the sake of argument) the conception of actual infinity implicit in the construction. I think that the answer is "All of it!" The facts used in the reasoning are all fairly elementary: e.g. there is a finite distance between any two distinct points. In particular, nothing special about the structure of the reals is used. If the construction were applied to the rational line, it would still generate an infinite partition; we would simply come up with the empty set more often. There is one slightly sticky issue regarding the foregoing construction, and that regards the seemingly trivial issue of where to throw the midpoint. One might argue that the line is not really divided into two equal parts if the midpoint is thrown on one side or the other. However, if the midpoint is put on both sides there is no partition. Such questions were, in fact, discussed. This one is raised for a slightly different purpose in the pseudo-Aristotelian treatise, 'Concerning Indivisible Lines.'[4] The objection could be met by modifying the construction. We could, at each stage, divide the line segment into three parts: the midpoint and two intervals. At the cost of some complication, Zeno's argument could still be carried through. Returning to the discussion in 'On Generation and Corruption':

> But suppose that, as the body is being divided, a minute section – a piece of sawdust, as it were – is extracted, and that in this sense a body 'comes away' from the magnitude, evading the division. Even then the same argument applies ($316^a 34 – 316_b 3$).

Zeno's Paradox of Measure

Suppose that the line segment is composed of an infinite number of parts. Zeno claims that this leads to absurdity in the following way:

(I) Either the parts all have zero magnitude or they all have positive magnitude.

(II) If they have zero magnitude, the line segment will have zero magnitude, since the magnitude of the whole is the sum of the magnitudes of its parts.

(III) If they have positive magnitude, then the line segment will have infinite magnitude, for the same reason.

(I) is never explicitly stated, but it is certainly an implicit supposition of the argument. In the first place, it assumes that the parts in question are the sorts of things which *have* magnitude; that questions of magnitude meaningfully apply. Call this the assumption of *Measurability*. It is challenged by Aristotle. There is something more being assumed. There would be no paradox if an infinite number of the parts had zero magnitude and a finite number had appropriate positive magnitudes. This possibility is ruled out. On what basis? One might argue that if the infinite partition of the line segment is generated by bisection, as discussed in the last section, then each part should have *equal* magnitude because of the equality in magnitude of all members of the finite partitions at each stage of the process which generates the infinite partition.

Call this the assumption of *Invariance*. The assumptions of *Measurability* and *Invariance* legitimize (I) in a plausible way, and do a bit more as well. Invariance will enter again later in the reasoning.

(II) and (III) share the assumption that the principle that the magnitude of the whole is the sum of the magnitudes of its parts continues to hold good when we have a partition of the whole into an infinite number of parts. This requires for its intelligibility that some general sense be given to the notion of the sum of an infinite number of magnitudes. We have no such definition from Zeno. We could, of course, *supply* one adequate to his purposes. Let S be an infinite set of magnitudes, and let S^* be the set of *finite* sums of magnitudes in S. A real number is an *upper bound* for S^* if and only if it is greater than or equal to every member of S^*. Let the *sum* of S be defined as the *least upper bound* of S^* if a real least upper bound exists, and as infinity otherwise. I will call this the principle of *Ultra-Additivity*. Zeno probably simply thought that *any* reasonable principle relating the magnitude of the whole to that of an infinite partition of it would have to satisfy (II) and (III). Nevertheless, it may be useful to have an explicit formation of such a principle at hand while considering the argument.

On the principle of Ultra-Additivity, (II) is perfectly correct. From a modern viewpoint, we would say so much the worse for the principle. This does not appear to be a line that was taken in ancient times. Neither the school of Plato, nor that of Aristotle, nor the Atomists appear to have challenged (II).

At first glance, (III) may appear to be a mathematical blunder. Does Zeno really believe that any infinite sum of finite magnitudes is infinite? His own paradoxes of the Dichotomy, and of Achilles and the Tortoise provide a counterexample, a fact that did not escape Aristotle.[5] However, if Zeno is here presupposing that he has in hand an *invariant* partition by means of the previously discussed construction, there is no blunder. As we noted, such an argument may already be required for (I). If so, it costs no more to use it twice.

There is a more delicate question to be raised about (III). What if the magnitudes involved were infinitesimal? Then (III) could fail. Infinitesimals are ruled out by the *Axiom of Archimedes* (probably originated by Eudoxus). This axiom entails that for any quantity, e, no matter how small, and any integer m, no matter how large, there is an integer n such that n times e is greater than m. If the magnitudes under consideration are Archimedean, then the relevant S^* will have *no* upper bounds, and (III) will be justified.

The question of who, if anyone, at this time held a theory of infinitesimal magnitudes is a matter of controversy. It is of some interest that Cajori takes one of Zeno's fragments to be an ironic dismissal of the theory of infinitesimal magnitudes: "Simplicius reports Zeno as saying: 'That which when added to another does not make it greater, and being taken away from another does not make it less, is nothing'. According to this, the denial of the existence of the infinitesimal goes back to Zeno" (Cajori, 1919, p. 51). However, it should be noted that Cajori's interpretation of this problematic fragment is not one favored by most classicists.[6] However this may be, it remains that (III) requires the assumption of the Archimedean axiom.

Zeno has shown that the supposition of all the following leads to paradox: the line segment (or more generally any body with positive magnitude) can be partitioned into an infinite number of parts such that: (I) the concept of Magnitude applies to the parts (Measurability); (II) the parts have equal magnitude (Invariance); (III) there are no infinitesimal magnitudes (Archimedean Axiom); (IV) the magnitude of the whole is the sum of the magnitudes of the parts in the sense given (Ultra-Additivity). The argument as we have reconstructed it contains no fallacy. A consistent theory of non-trivial magnitudes must give up some part of the foregoing. That Zeno had raised a genuine problem seems to have been well enough understood by his contemporaries. Different solutions were explored by different schools.

Aristotle's Answer to Zeno

Aristotle blocks the paradox of measure at two places. As we have already noted, he denies that infinite divisibility allows the construction of an infinite partition. Aristotle will follow the construction to any finite level n, but not to level ω. Thus: "A thing is infinite only potentially, i.e. the dividing of it can continue indefinitely ... " (Aristotle, 'On Generation and Corruption,' 318^a21. See also Aristotle's discussion in 317^a1–16 where he claims that the argument rests on a confused assumption that "point is 'immediately next' to point." It appears, however, that it is Aristotle himself who is confused here. As we have seen, the argument requires no such assumption, and it is nowhere to be found in Aristotle's lucid statement of the argument in 316^a1–316_b14.) Again, in the *Physics*, Bk. III Ch. 6, we have:

Now, as we have seen, magnitude is not actually infinite. But by division it is infinite. (There is no difficulty in refuting the theory of indivisible lines.) The alternative then remains that the infinite has a potential existence.

But the phrase 'potential existence' is ambiguous. When we speak of the potential existence of a statue we mean that there will be an actual statue. It is not so with the infinite. There will not be an actual infinite (206^a16–20).

Now, as Fränkel (1942) and Owen (1957–8) point out, Zeno has available a devastating *argumentum ad hominem* for anyone who, like Aristotle, will grant each finite n but will resist the move to level ω on the grounds that it presupposes the completed actual infinity of all the n stages. That is: "How does Achilles catch the Tortoise?", "How can I move from here to there?" Achilles does not catch the Tortoise at any finite stage of the process that Zeno describes; likewise with respect to the Dichotomy. Let us make the connection very explicit. Let me walk a unit distance. First, I walk half the distance, then half the remaining distance, etc. At each stage consider the set of points containing my (center of mass) location, the endpoint of the interval, and all the points in between these sets form the chain: [0, 1], [1/2, 1], [3/4, 1] .... I do not arrive at the endpoint at any finite stage of the process, but only at the ωth stage where the set in question contains only one member – the endpoint.

Could the possibility of this line of argument have escaped Aristotle? He comes perilously close to conceding its validity later in the same discussion in the *Physics* (206_b3–10):

In a way the infinite by addition is the same as the infinite by way of division. In a finite magnitude, the infinite by addition comes about in a way inverse to that of the other. For in proportion as we see division going on, in the same proportion we see addition being made to what is already marked off. For if we take a determinate part of a finite magnitude and add another part *determined* by the same ratio (not taking the same amount of the original whole) and so on, we shall not traverse the given magnitude.

But we *do* traverse a given magnitude. Motion is real enough for Aristotle. Zeno appears to have demonstrated from Aristotelian premises, the existence of actual completed infinity. (Indeed the method of argument used in the dichotomy can be adapted to show how wiggling one's finger instantiates all sorts of countable ordinals.) Aristotle appears to grant the point in Bk. VI, Ch. 2 of the *Physics*: "Hence it is that Zeno's argument makes a false assumption when it asserts that you cannot traverse an infinite number of things one by one, in a finite time ... and the contact with the infinities is made *by means of moments not finite but infinite in number*" (*ibid.*, 233^a13–31, emphasis mine), only to retract it in a curious discussion in Bk. VIII, Ch. 8 263^a4–263_b9. After a discussion of what happens to the midpoint when a line segment is bisected, he returns to the question of potential vs. actual infinity:

Therefore to question whether it is possible to pass through an infinite number of units either of time or of distance we must reply that in a sense it is and in a sense it is not. If the units are actual, it is not possible: If they are potential it is possible. For in the course of a continuous motion the traveler has traversed an infinite number of units in an accidental sense but not in an unqualified sense: For though it is an accidental characteristic of the distance to be an infinite number of half-distances, this is not its real and essential character (*ibid*., 263_b4–9).

Aristotle has a second related reply to the paradox: he denies *Measurability*. Points are not to be thought of as *parts* of lines at all, and thus are not sorts of things that have magnitude. The things that qualify as genuine parts of a line segment are non-degenerate subsegments. This is argued in Book VI of the *Physics* on the basis of Aristotle's theory of continuity as a kind of contiguity: "nothing that is continuous can be composed of indivisibles: e.g. a line cannot be composed of points, the line being continuous and the point indivisible" (*ibid*., 231^a24–25). (Aristotle holds that "things are continuous if their extremities are one." Presumably an example would be the closed intervals [0, 1/2] and [1/2, 1] which are continuous and make up the interval [0, 1].[7] Aristotle does not seem to have the concept of an open interval.)

Aristotle has another reason for holding that points are not measurable parts of the line, and that is that aside from the objections stated here, he appears to hold that Zeno's argument is valid. In 'On Generation and Corruption,' (317^a1–9) where he explains what *the* fallacy of the argument is, he points only to the argument that the line can be thought of as composed of points.

Whatever Aristotle's reasons for holding that points are not legitimate parts of lines or bodies, the idea has been attractive for a number of thinkers, e.g. Whitehead. Points can, in a sense, be eliminated in favor of intervals, of course. We can take intervals as basic, and taking our cue from Zeno's construction, let chains of intervals stand for the points respectively that are the unit sets of their intersections. Points then re-emerge as logical reconstructions. (For a general account of this sort of approach see Tarski, 1956.) Such points would naturally be assigned as the limit of the lengths of the members of the chain, i.e. length zero. So while this approach retains a faint Aristotelian flavor, it partakes of strongly non-Aristotelian elements as well.

Aristotle's answer to Zeno's paradox of measure shows the temperament of the empirical scientist rather than that of the mathematician. His response involves a theory of some subtlety, but that theory is mathematically conservative if not reactionary. With regard to both infinity and measurability his instinct was to restrict the subject to a narrow and safe domain rather than to explore uncharted waters.

Indivisible Magnitudes

There are at least two schools that postulated some kind of indivisible magnitudes. Aristotle, in 'On Generation and Corruption,' cites what is essentially Zeno's argument as a motivation for the adoption of indivisible magnitudes by the Atomists. A doctrine of indivisible magnitudes was also held in the Platonic academy. Aristotle ascribes it to Plato himself:

Further, from what principle will the presence of *points* in the line be derived? Plato even used to object to this class of things as being a geometrical fiction. He gave the name of the principle of the line – and this he often posited – to the indivisible lines (*Metaphysics*, 992^a19–21).

It is commonly ascribed to his student Xenocrates, and the pseudo-Aristotelian polemic 'Concerning Indivisible Lines' is thought to be directed at this doctrine. Details of either theory are hard to come by. The evidence available is sparse and mostly circumstantial.

One wonders, in each case, whether the indivisible magnitudes are meant to be infinitesimal or finite. Each alternative would block Zeno's argument, but they would block it in different ways. Finite indivisible magnitudes would block the construction of the partition of a thing of finite magnitude into an infinite number of equal parts. Infinitesimal indivisible magnitudes are consistent with the existence of an infinite partition, but allow for the possibility that an infinite number of positive magnitudes add up to a finite magnitude. Some commentators attribute a doctrine of infinitesimals to both schools (not to mention the Pythagoreans and Anaxagoras) but the evidence seems far from conclusive.

The Archimedean manuscript 'On Method,' discovered by Heiberg in 1906, shows that Archimedes used infinitesimals in a method of discovery, although he did not consider that infinitesimal methods provided a strict proof. Archimedes attributed the discovery of the volume of the pyramid and of the cone to Democritus, but says that he did not prove it rigorously. Boyer (1968, pp. 88–89) remarks: "This creates a puzzle, for if Democritus added anything to the Egyptian knowledge here, it must have been some sort of demonstration, albeit inadequate." Boyer suggests that this may point to a theory of infinitesimals. In the light of this speculation the following fragment of Democritus becomes even more tantalizing:

If a cone is cut along a plane parallel to its base, what must we think of the surfaces of the two segments – that they are all equal or unequal? If they are unequal, they will make the cone uneven, with many step-like indentations and roughnesses: If they are equal, then the segments will be equal and cone will turn out to have the same properties as the cylinder, being composed of equal and not of unequal circles – which is quite absurd [from Plutarch, *De communibus notitiis*, tr. Furley in Furley (1967), p. 100].

But whatever the truth of the matter is regarding Democritus, it appears clear that his follower Epicurus did not believe in infinitesimal magnitudes. His letter to Herodotus contains a passage which is inconsistent with that doctrine:

For when someone says that there are infinite parts in something, however small they may be, it is impossible to see how this can still be finite in size; for obviously the infinite parts must be of *some* size, and whatever size they may happen to be, the size (of the total) would be infinite [tr. Furley in Furley (1967), p. 14].

It is, of course possible that Epicurus modified the doctrine of Democritus, but in that case we might expect to find a more explicit discussion of the whole issue.

The main evidence regarding Xenocrates' theory is the treatise which was presumably written to refute it: 'Concerning Indivisible Lines.' Unfortunately the evidence is not univocal. Owen reads the treatise as directed against finite indivisible magnitudes:

It is not certain whether the proponents of this theory [of indivisible lines] thought that every measurable distance contained a finite or an infinite number of such distances. An argument for thinking the former is that this is assumed in the fourth-century polemic *On Indivisible Lines*. An argument for thinking the contrary is that the theory was held at a time when the difficulties of incommensurable lines were fully realized. It was a commonplace that the side and diagonal of a square cannot both be *finite* multiples of any unit of length whatsoever [Owen (1957–8), p. 150].

On the other hand, Boyer does not hesitate to interpret the treatise as directed against a theory of infinitesimals:

The thesis of the treatise ['On Indivisible Lines'] is that the doctrine of indivisibles espoused by Xenocrates . . . is untenable. The indivisible, or fixed infinitesimal of length or area or volume, has fascinated men of many ages; Xenocrates thought that this notion would resolve the paradoxes, such as those of Zeno, that plagued mathematical and philosophical thought [Boyer (1968), p. 108].

Indeed, the treatise is such a scattershot affair that it is hard to detect the target. Perhaps there was more than one target. Otherwise the author was hopelessly confused.

In the beginning of the treatise, the author lists the arguments which the proponents of indivisible lines use to support their doctrine. The fourth argument is only consistent with the indivisible magnitudes being finite rather than infinitesimal:

Again, Zeno's argument proves that there must be simple magnitudes. For the body, which is moving along a line must reach the half-way point before it reaches the end. And since there is always a half-way point in any 'stretch' which is not simple, motion – unless there be simple magnitudes – involves that the moving body touches successively one-by-one an infinite number of points in a finite time which is impossible ('Concerning Indivisible Lines,' 968^a18–21).

But later in the treatise, we have already noted the passage which appears to be inconsistent with the indivisible lines being of finite magnitude:

Further, the addition of the line will not (on the theory) make the whole line any longer than the original line to which the addition was made: for simples will not, when added together, produce an increased total magnitude (*Ibid.*, 970^a21–23. See also 970_b23–25, 972^a12–14; Aristotle, *Physics*, 220^a15–20; 263^a4–9; *On Generation and Corruption*, 316^a24–34).

Perhaps it is best just to say that both sorts of theory were in the air, without trying to be too positive about who held which theory.

Neither of these theories leaves Zeno bereft of argument. The theory of infinitesimals was not put on a firm foundation until the work of Abraham Robinson in this century. It would not be much of a problem to embarrass whatever preliminary ideas there were about them at the time. Indeed, if the theory in question *did* claim that the addition of a single infinitesimal magnitude would not make the line segment any longer, then Zeno had already made the reply: "If when it is taken away, the other thing is to be no smaller, and is to be no bigger when it is added, it is clear that what was added or taken away was nothing." Owen suggests that Zeno's paradox of the *Stadium* is also directed against infinitesimal magnitudes. His interpretation is only possible, however, if the theory of infinitesimals in question has the simple parts of the line discretely ordered. Any respectable theory of infinitesimals would have the simple parts of infinitesimal magnitude densely ordered. But since we are not in possession of the theory of infinitesimals in question (if there *was* one in question) we do not know how respectable it was.

However that may be, both the *Arrow* and the *Stadium* raise difficulties for an account of motion on a theory of *finite* indivisible magnitudes. The flying arrow moves but does not move at any instant. Motion becomes "first here, then there." Average velocity may make sense but instantaneous velocity does not. The Stadium shows us how considerations of relative motion almost *force* us into infinite divisibility. Even before considering relative motion there is this problem: If something is travelling a space unit for every two time units, where is it after one time unit? Conversely, if it is travelling two space units per time unit, how much time has elapsed after it travels one space unit? (Aristotle advances such arguments in the *Physics*, Bk. VI, Ch. 2, 232_b20 ff.) A theory of finite indivisible magnitudes might reject such questions, or failing that, might try to get by with a theory that only allowed rest and motion of one *speed*, i.e. that of one time unit per space unit. Zeno's paradox of the *Stadium* shows that this strategy does not escape the problem. By considering one series of bodies at rest, and two having unit speed but opposite direction, the embarrassing questions can be asked again in terms of relative motion.

Aristotle did not hesitate to use the Arrow against indivisible magnitudes. He argues that on the rival theory " . . . the motion will consist not of motions but of starts, and will take place by a thing's having completed a motion without being in motion . . . So it would be possible for a thing to have completed a walk without ever walking . . . " (*Physics*, Bk. VI, Ch. 1, 232^a9–11). Epicurus took the point. Furley reports the following comment of Themistius:

> But our clever friend Epicurus is not ashamed to use a remedy more severe than the disease – and this inspite of Aristotle's demonstration of the visciousness of the argument. The moving object, he says, *moves* over the whole distance, but of each of the indivisible units of which the whole is composed it does not move but has *moved* [tr. Furley in Furley (1967), p. 113].

With regard to degree of motion, Epicurus again accedes to Aristotle. He actually maintains that all atoms *do* move through the void with the same speed [though not the same direction – see Furley (1967), p. 121 ff.]. That the *Stadium* shows that little or nothing is gained by this desperate move, appears to have escaped Epicurus as it did Aristotle.

Conclusion of Section I

We have seen that Zeno's paradox of measure rests on the following premises:

(I) *Partition*: the line segment can be partitioned into an infinite number of parts such that:
(II) *Measurability*: the concept of magnitude applies to the parts.
(III) *Invariance*: the parts all have equal positive magnitude, or zero magnitude.
(IV) *Archimedean Axiom*: there are no infinitesimal magnitudes.
(V) *Ultra-Additivity*: the magnitude of the whole is the sum of the magnitudes of the parts in the sense given.

Ancient attempts to answer Zeno focused largely on (I) and (II). Doctrines of finite indivisible magnitudes (certainly Epicurus and probably Democritus and Leucippus) rejected (I). Aristotle rejected (I) and (II). It is possible that a doctrine of infinitesimal indivisible magnitudes was also current (possibly held by Xenocrates, possibly by Democritus) which rejected (IV). (III) could have also been challenged by a holder of a doctrine of infinitesimal magnitudes. (V), Ultra-Additivity, appears to have been accepted without question by every party to the dispute. It is ironic that it is just here that the standard modern theory of measure finds the fallacy.

II. POST CANTOR

Measure According to Peano and Jordan, Borel, and Lebesgue

It is no accident that Zeno was first taken seriously in the modern era by mathematicians (see Tannery, 1885) at a time when problems in the theory of integration were leading to the development of measure theory. The concept of measurability was introduced (putting aside Aristotle's response to Zeno) by Peano in 1883 and generalized by Jordan in 1892 (for details see Hawkins, 1970). In discussing areas in the plane, Peano considers (I) the class of polygons which contain the region in question and (II) the class of polygons which are contained in the given region. The area of the region should be less than or equal to the areas of the polygons in the first class and greater than or equal to the areas of the polygons in the second class. If these conditions determine a unique number [i.e. if the greatest lower bound of the areas of polygons in class (I) equals the least upper bound of polygons in class (II)], then that is the area of the region. If not "then the concept of area would not apply in this case" (from Peano, 1883; quoted in Hawkins, 1970, p. 87).

Thus, on the line, an interval [a, b] is assigned its length, b – a, as measure.[8] This includes points which as degenerate intervals [a, a] are assigned measure zero. These measures are fundamental, and the concept of measure is extended to other point sets as follows. Consider finite sets of intervals which *cover* the set of points in question in that it is contained in their union. Associate with each such covering set the sum of the lengths of intervals in it. The greatest lower bound of these numbers is called the *outer content* of the set. Working from the other side, consider finite sets of non-overlapping (pairwise disjoint) intervals whose union is contained in the set in question. Associate with each such set the sum of the lengths of its members. The least upper bound of these numbers is called the *inner content* of the set. If the outer and inner content of a point set are equal, then the set is *measurable in the sense of Peano and Jordan* and that number is its measure. If not, then the set is not measurable – the concept of measure simply does not apply.

Jordan showed that measure, so defined, is *finitely additive*. That is, if each of a finite collection of mutually disjoint sets is measurable, then their union is also and its measure is the sum of theirs. The appropriate principle of additivity is not assumed in the definition, but rather proved from the definition; and it is a rather modest kind of additivity. The stronger principle of *countable* additivity fails for Peano–Jordan measure. The union of a denumerable collection of measurable sets may not itself be measurable. For instance, the set of rational points in [0, 1] is not Peano–Jordan measurable. Its outer content is 1, while its inner content is 0. Yet, as Cantor had shown, it is the union of a denumerable collection of unit sets.

The basic ideas of Peano–Jordan measure could have been introduced in Aristotle's time. They depend only on finite sums of intervals. The restriction to finite additivity allows a rich theory of measurable sets. Not every set of points becomes measurable, but the assignment of measure zero to unit sets of points causes no difficulties. Finite sets of points must have measure zero, but the Greek geometers knew well enough that the line was not exhausted by any finite set of points.

Borel took a rather different approach to measure and measurability in 1898. Borel *constructs* the Borel measurable sets out of the intervals by finite and denumerable set theoretic operations, and defines their measure by postulating a stronger form of additivity: i.e. countable additivity. A collection of sets is called a *sigma-algebra* if it is closed under countable union and intersection, and complementation. The Borel-measurable sets on the line segment can be defined as the smallest sigma-algebra of point sets containing the open intervals. Intervals have their length as their measure. Measure

is taken to be countably additive (or sigma additive). That is, a countable union of mutually disjoint intervals has as its measure the infinite sum of the lengths of the intervals. Sigma additivity can be thought of as the restriction of the fancied principle of ultra-additivity of Part I to denumerable collections. Any denumerable set of points, e.g. the rationals in [0, 1], has Borel measure zero since it is a countable union of singletons each of which has measure zero. Since [0, 1] has measure 1, the set of irrational points in [0, 1] has measure 1. As Grünbaum has emphasized, this causes no problems because this set had been shown by Cantor to be uncountable. For this reason, Borel's theory of measure was only conceivable after Cantor's fundamental investigations of infinite cardinality.

This is not to suggest that only countable point sets have measure zero. Consider the famous Cantor ternary set. It can be constructed by starting with [0, 1] and then removing the middle third open interval (1/3, 2/3). Thus we have at stage 1 of the construction the points in [0, 1/3] and [2/3, 1]. To move from stage n of the construction to stage $n + 1$ we delete the middle open thirds of the closed intervals of stage n. The intersection of the sets at finite stages of the construction is Cantor's ternary set. It has Borel measure zero since we started with a set of measure 1 from which we have subtracted a set of points which by countable additivity has measure one. Alternatively, it has measure zero in the sense of Peano and Jordan, since each stage n in the process of construction provides a finite covering with measure $(2/3)n$, the outer content of the Cantor set is zero. Nevertheless the Cantor set is non-denumerable. The interval [0, 1] can be mapped 1-to-1 into the Cantor set. Remember Zeno's construction by infinite bisection of the line in Section I of this paper. For each point on the line, at each bisection it was either on the left or the right. The intersection of each chain, in Zeno's construction, contained at most one point. So each point on the line corresponds to a unique infinite sequence of 'left' and 'right'. Applying such a sequence to the stages in the construction of the Cantor set, we select the indicated left or right third which remains, so we have corresponding to each point in the original interval a unique chain of closed intervals. The intersection of such a chain must be non-empty by the Heine-Borel theorem. The chains are constructed in such a way that no point can be in the intersection of two such chains. So each point on the original line corresponds to a unique point in the Cantor set.

Borel's bold move to countable additivity was not received without some qualms in the contemporary mathematical community. In a report on the theory of sets published in 1900, Schoenflies was critical of this as well as other aspects of Borel's theory of measure. With regard to countable additivity, he writes that "the question of whether a property is extendable from finite to infinite sums cannot be settled by positing it but requires further investigation" (quoted in Hawkins, 1970, p. 107).

The further investigations were successfully carried out by Lebesgue in 1902. Lebesgue generalized the notions of inner and outer content of Peano and Jordan in such a way that the countable additivity of measure could be demonstrated; Lebesgue's definition of *outer measure* considers *denumerable* coverings. For each countable covering consider the limiting sum of the lengths of its constituent (open) intervals. The greatest lower bound of these numbers is the *outer measure* of the set in question. (Notice that the Lebesgue outer measure of the set of rationals in [0, 1] is 0.) For the inner measure of a bounded set, S, consider the closed intervals [a, b] which contain it. For each take its length, b – a, minus the outer measure of the set of points in it which are not in S. We can define the *inner measure* of S as the least upper bound of these numbers. (Of course, these numbers are all really the same. Any closed interval containing S will give the same result.)

Lebesgue was able to prove on the basis of these definitions, that the Lebesgue-measurable sets include both the Borel measurable sets and the sets measurable in the sense of Peano and Jordan; that Lebesgue measure agrees with each of these measures on the sets for which those measures were defined, and Lebesgue measure is countably additive. Furthermore, he showed that Lebesgue measure has the intuitively correct property of *translation invariance*. For a set S, and a real number a, let the set $S + a$ contain just the points $x + a$ for every x in S. The Lebesgue measure of any measurable set S, equals the measure of $S + a$ for any real number a.

Lebesgue's theory showed how the virtues of earlier theories could be combined and extended to provide an intuitive treatment for a very rich domain of measurable sets. In fact, at the time, it was not immediately apparent whether there were any bounded sets which were not measurable in the sense of Lebesgue.

The Vitali Paradox

In 1905 Vitali produced the first example of a non-Lebesgue measurable set. The argument is in many ways strikingly similar to that used in Zeno's paradox of measure. Since Lebesgue measure is only countably additive, rather than ultra-additive, one following the path of Zeno would have to seek

a *countable* partition of the unit line segment into parts which by some symmetry consideration should have the same magnitude. With such a partition in hand, he could argue that if the members of the partition have zero measure, then the unit interval must have zero measure; if they have equal positive measure, the unit interval must have infinite measure. Both alternatives contradict the fact that the unit interval has Lebesgue measure one, so the members of the partition are not Lebesgue measurable.

Vitali found such a partition. To simplify matters slightly, we will construct the partition of the half-open interval [0,1). We can visualize this as wrapped around to form a unit circle. The relevant symmetry property of Lebesgue measure was mentioned in the preceding section. It is *translation-invariance*. Translation invariance implies translation invariance modulo 1, which in terms of our visualization means that if any Lebesgue measurable set of points is displaced a fixed distance around the circle, the resulting set will have the same Lebesgue measure. Consider the equivalence relation: x – y is rational. This partitions [0, 1) into equivalence classes. Choose one member from each of these classes to form the choice set C. For each rational, r, in [0, 1) let C_r be the set gotten by adding (modulo 1) r to each member of C (i.e. by displacing C the distance r around the circle). The C_rs form a denumerable partition of [0, 1). Any one can be gotten by translation from any other. Since Lebesgue measure is translation-invariant, if they are Lebesgue measurable, they have the same measure. If so, the measure of [0, 1) must be either 0 or infinity. So the C_rs are not Lebesgue measurable. Such non-measurable sets are ubiquitous. It can be shown that every Lebesgue measurable set with non-zero measure contains a non-measurable set. Zeno would have been delighted.

Vitali's construction requires stronger mathematical methods than Zeno's. The crucial step involves the axiom of choice. This proves to be essential in the construction of a non-measurable set (Solovay, 1970).

The only facts about Lebesgue measure used in Vitali's argument other than translation invariance are that it is *countably additive* and *real valued* (the latter being used for the Archimedean property of the real numbers). Thus, the argument establishes a more general result: any translation-invariant, countably additive, real-valued measure defined on all the subsets of [0, 1)[9] must give [0, 1) either infinite measure or measure zero.

Must we, with Aristotle, concede that intervals (areas, volumes) of positive magnitude are made up of parts to which the concept of magnitude does not apply? Or can we plausibly weaken the foregoing set of three conditions which generate the Vitali paradox?

Finite Additivity and Non-Archimedean Measure

Lebesgue measure escaped Zeno's paradox by virtue of a weaker form of additivity. This suggests that a weakening of countable additivity to finite additivity might allow us to define a finitely additive measure on richer domain of sets. Of course, such a possibility would only be of interest if some of the virtues of Lebesgue measure could be retained; e.g. we would like each interval to have its length as its measure.

In fact, we can have this and more. There is a finitely additive, real-valued translation invariant measure defined on *all* subsets of [0, 1],[10] which agrees with Lebesgue measure on all the Lebesgue measurable sets.[11]

Returning to Vitali's example, it is clear that such a measure must give the sets C_r measure zero, for if it gave them positive measure, *finite* additivity and translation invariance would contradict the measure of [0, 1) being one. The C_rs can thus be accommodated by a finitely additive measure in the way in which the singletons were accommodated by Lebesgue measure; they have measure zero but the additivity properties of the measure are not strong enough for that to cause problems.

Some philosophers may, despite all of this, feel nagging Zenonian intuitions to the effect that a whole of positive magnitude simply should not be made up parts of measure zero. This is the intuition that measure should be *regular*; that only the null set should receive measure zero. We have seen that not even a *finitely* additive translation invariant measure can accommodate this intuition if it is real valued. But what if the values that the measure takes on lie in a domain which is non-Archimedean? Couldn't we get away with giving both the singletons and the C_rs infinitesimal measure in some way in which everything works out nicely?

Such speculations may be very old, but it has only been possible to give them substance since Abraham Robinson's creation of non-standard analysis (Robinson, 1966). Leibniz thought of infinitesimals as ideal elements which nevertheless obey the same laws as the numbers. But which laws? The answer cannot be "All" in too strong a sense; otherwise we would not be able to distinguish a theory which admits infinitesimals from one which doesn't. This question had to wait for the development of model theory for its proper answer. Robinson showed how a non-standard model of analysis could incorporate infinitesimals, which consequently must obey the *first-order laws* which govern the real numbers.

The crucial logical property of first-order languages that Robinson's construction uses is compactness: if a set of sentences is such that every finite subset of it has a model, then the whole set in question has a model. Compactness of first-order languages depends on their limited logical resources: the logical constants being limited to truth functions, identity, and first order quantifiers and their sentences being of only finite length. It does not depend on the languages being denumerable. Thus we could (and will) imagine first order languages with names for every real number, which are nevertheless compact. Compactness fails for second order logic given the 'natural' interpretation of second order quantifiers having as their domain the power set of the domain of the first order quantifiers. However, if we allow Henkin's *general* models in which higher order quantifiers are allowed to have as their domain subsets of their natural domain, higher order quantification theory is also compact.

Here then, is how we get a non-standard model of analysis which contains infinitesimal elements: Consider a rich first-order language which for every real number, r, contains a name o_r; a relational symbol for every relation on the reals; and an operation symbol for every operation on the reals. Let the theory ANALYSIS consist of all the true sentences of this language, and consider the theory which is the union of ANALYSIS with the set of all sentences of the form $o_r < y$ for each real r. Each finite subset of this theory has a model in the reals, so by compactness this theory does too. This is a non-standard model of the reals. The function which maps each real, r, onto $o_r{}^*$, the denotation in the non-standard model of its name, is an isomorphism. Each non-standard model contains an isomorphic copy of the reals. Working within the non-standard model, we will simply call these the *standard reals*. The denotation of the less-than relation totally orders the non-standard reals since the axioms of total order are first-order. According to that order, the element which the model assigns as the denotation of y is a infinite element; it is greater than any of the standard reals. There is a first order sentence which says that every number has a reciprocal and one which says that if x is greater that y, then the reciprocal of x is less than the reciprocal of y. Since the model makes these sentences true, there must be an element of the model that is the reciprocal of the infinite element and less than any positive standard real. This is an infinitesimal element. A great deal of knowledge about the structure of the infinitesimals follows from the fact that they obey all first order generalizations about the reals.

The question as to how such infinitesimals can be incorporated into non-standard measure theory is a bit more complicated, involving non-standard (general) models for a higher order language of analysis. (For details see Bernstein and Wattenberg, 1969.) They show that one can construct a measure defined for all subsets of the unit interval, which takes its values in a non-standard of the reals, which is *finitely additive, translation invariant up to an infinitesimal* which is *infinitesimally close to Lebesgue measure* on the Lebesgue measurable sets, and which is *regular* (i.e. only the null set gets measure zero). The Vitali sets of the last section, and the sets containing exactly one point will then both have infinitesimal measure.

It can be shown that in non-standard models of analysis every non-standard real is infinitesimally close to a unique standard real. Call the second the *standard part* of the first. Then if we have a non-standard measure of the kind described here, and derive a real-valued measure by considering only the standard parts of the values assigned by the non-standard measure, we get the sort of measure discussed at the beginning of this section: a real-valued, finitely additive, translation invariant measure defined on all subsets of [0, 1] which agrees with Lebesgue measure on the Lebesgue-measurable sets. What we gain by allowing our measure to take values in richer range – the non-standard reals – is *regularity*.

The Hausdorff Paradox

We seem to have finally seen how to get rid of non-measurability. Banach showed that at the cost of weakening additivity to finite additivity on the non-Lebesgue-measurable sets, we can make every bounded set of points on the real line measurable. This does not quite lay Zeno to rest, for he was ultimately concerned with magnitudes of volumes in three-dimensional space. We have been confining ourselves to one dimension for the sake of simplicity. To complete the story, we should show that Banach's result can be extended to three-dimensional space. It is not so. A construction due to Hausdorff (1914) and further generalized by Banach and Tarski (1924) shows that one cannot in three and higher dimensional Euclidean spaces have a finitely-additive measure, which assigns the unit cube measure 1, assigns congruent point sets equal measure, and assigns a measure to all subsets of the unit cube.

Here the appropriate invariance property is congruence-invariance. Points here are to be thought of as triples of real numbers. The *Euclidean distance* between two points, (x, y, z) and (x', y', z'), is given by the Pythagorean formula:

$$[(x-x')^2 + (y-y')^2 + (z-z')^2]^{\frac{1}{2}}.$$

Two sets of points in Euclidean three-dimensional space are congruent just

in case there is a 1-to-1 function mapping the one onto the other which preserves Euclidean distance (i.e. the distance between any two points in the first set is equal to the distance between their images in the second set). Congruence invariance in one dimension is just translation invariance. The theory of Lebesgue measure for n-dimensional Euclidean space, developed analogously to the theory for one dimension, has the consequence that Lebesgue measure is congruence-invariant on the Lebesgue measurable sets. Banach actually showed that a finitely additive congruence-invariant extension of Lebesgue measure to all bounded sets is possible in both one- and two-dimensional Euclidean space. It is only in Euclidean spaces of three and higher dimensions where the theorem fails.

In an extended note to *Grundzüge der Mengenlehre* (1914) headed 'Unsolvability of the Measure Problem,' Hausdorff sets out to show that it is impossible to assign to all point sets on the surface of a sphere a finitely additive, congruence-invariant measure which assigns the whole surface a positive measure. To this end, he proves the following theorem:

> The spherical surface, K, can be decomposed into disjoint sets: A, B, C, Q, where Q is countable; A, B, C are congruent to each other; the union of B and C is congruent with each of the sets A, B, C.

Since Hausdorff is here considering real-valued measure, congruence invariance together with a finite measure for the surface entails that each countable point set has measure zero. (For one can by appropriate choice of rotations generate an infinite number of disjoint congruent point sets to any given denumerable point set. If the given set has positive measure, the surface of the sphere by finite additivity could not have finite measure.) Thus, under the stated assumptions, the measure of the surface, $m(K)$, would equal $m(A) + m(B) + m(C)$. Since A, B, and C are congruent with each other, $m(A) = 1/3\ m(K)$. Since A is congruent with $B \cup C$, $m(A) = 1/2\ m(K)$.

Hausdorff's theorem again depends on the axiom of choice (as does Banach's positive result for 1 and 2 dimensions). Hausdorff works with a group of rotations about two appropriately chosen[12] axes; the group generated by 1/2 rotation about the first axis, ϕ, and 1/3 rotation about the second, ψ. Hausdorff shows how this group of rotations can be decomposed into three disjoint sets: $G = A \cup B \cup C$, such that $A \cdot \phi = B \cup C$; $A \cdot \psi = B$; $A \cdot \psi^2 = C$.[13] Let Q be the countable set of fixed points of members of G. The set of points on the surface less this denumerable set, $S - Q$, is the disjoint union of the orbits of the group G. The axiom of choice comes into the picture to assure the existence of a choice set, M, containing exactly one member from each orbit. $S - Q$ consists of the union of the point sets that M is carried into by members of G. Let the *point* set A be the set of points that M is carried into by the rotations in the *set of rotations* A, likewise for B and C. Then A, B, C and Q are the requisite point sets for Hausdorff's theorem.

Hausdorff concludes: "A determination of measure for all bounded sets, which satisfies conditions . . . [congruence-invariance, unit cube has measure 1, finite additivity] is therefore impossible in three and higher dimensional Euclidean space, since otherwise it would also be possible on the sphere (where one would assign to a set on the sphere the volume of the corresponding conical body as its measure)."

The paradoxical results of Hausdorff and Vitali are analyzed and generalized in a celebrated paper of Banach and Tarski (1924). There they introduce the notion of *equivalence of sets of points by finite (and alternatively by denumerable) decomposition*. Two sets of points (in a metric space) are equivalent by finite decomposition iff there exist finite partitions $[p_1, \ldots p_n]$, $[q_1, \ldots q_n]$ of them respectively, whose respective members are congruent (p_1 congruent with q_1 & . . . & p_n congruent with q_n); analogously for equivalence by denumerable decomposition. Then generalizing Hausdorff's argument: "In a Euclidean space of $n \geqslant 3$ dimensions, two arbitrary sets, bounded and containing interior points (e.g. two spheres of different radius) are equivalent through finite decomposition" (Banach and Tarski, 1924, p. 244). In the form in which they develop it, Hausdorff's paradox is perhaps better known as the Banach–Tarski paradox. The analogous theorem holds for the surface of the sphere but fails for Euclidean spaces of 1 and 2 dimensions. For these spaces, however, we have a generalization of the Vitali paradox. For Euclidean spaces of dimension 1 and higher "two arbitrary sets (bounded or not) containing interior points are equivalent by *denumerable* decomposition" (Banach and Tarski, 1924, p. 244).

These rather surprising facts about *congruence* are at the heart of the restrictions on measurability that we have been discussing for the last three sections. They might be taken as calling into question the status of congruence-invariance as a desideratum for measure. Our intuitions in this regard are based on consideration c´ ar simpler point sets than the ones involved in the Vitali and Hausdorff paradoxes. Before even raising questions of measure, we see that our intuitions regarding congruence of simple bodies in three-dimensional Euclidean space cannot be projected to arbitrary point sets.

One can extend Lebesgue measure to a finitely additive, *non-congruence invariant* measure on all the bounded subsets of Euclidean three-dimensional space. So measurability of all bounded sets can be achieved, but at an unexpected cost. One might wonder, however, whether if one pays the price of giving up congruence-invariance, one can avoid weakening countable additivity to finite additivity. The Vitali paradox and its generalizations, after all, used congruence invariance essentially. Things, however, are not quite so simple. Non-measurability has roots that go deeper than the metric structure of the underlying space.

Non-measurable Sets Without Congruence Invariance

Let us recall, for a moment, Zeno's two principles from Section I. Suppose that a whole can be partitioned into an infinite number of parts. Then Zeno thought:

(I) If the parts had positive (real) magnitude, then the whole would have infinite magnitude.

(II) If the parts had zero magnitude, then the whole would have zero magnitude.

Let us by 'magnitude' understand a *countably additive* measure. Then (II) is correct for an infinite partition into a denumerable number of parts, but fails for partitions into a non-denumerably infinite number of parts. On the other hand, (I) can fail for a denumerable partition unless some extra assumptions about the magnitudes of the parts are present (e.g. that they must all be the same by some invariance argument). A fact that we have not taken explicit notice of yet, is that (I) holds without restriction if the infinite partition is non-denumerable. *If a set has finite measure, it can contain at most a denumerable infinity of disjoint sets of positive measure*. Consider any partition of the set in question. Consider the set of members of this partition with measure greater than or equal to 1/2. It must be finite. Otherwise by *finite* additivity of measure, the measure of S could not be finite. Likewise for the set of members of the partition with measure greater than or equal to $1/2^n$ and less than or equal to $1/2^{n-1}$, for each natural number n. Each member of the partition with non-zero measure is in one of these finite collections of sets. The number of such collections is denumerable, so the number of members of the original partition with positive measure is denumerable. Non-denumerable partitions make (I) true and (II) false;

denumerable partitions make (II) true [assuming countable additivity of the measure] and (I) false.

These facts make it possible to show, under the assumption of Cantor's continuum hypothesis, that there is no non-trivial countably additive measure on [0, 1] which gives all the unit point sets measure zero. The result is due to Banach and Kuratowski (1929) and was strengthened and generalized by Ulam (1930). No assumption of translation invariance is used; there is no appeal to metric considerations. Only the cardinality of the set in question plays a role. A set is of power aleph_1 iff it can be put into one-to-one correspondence with the ordinal numbers less than the first uncountable ordinal. The stated theorem is proved for arbitrary sets of power aleph_1. Cantor's continuum hypothesis enters to assure that [0, 1] is such a set.

I give the proof so that the reader can appreciate the Zenonian counterpoint: *Suppose that a countably additive (real-valued) measure is defined on a set, Z, of cardinality* aleph_1, *such that every one element subset receives measure zero. Then the measure of Z must be either infinite or zero.* Since Z is, by hypothesis, of power aleph_1, there is a well-ordering such that each element of Z is preceded by only countably many elements, i.e. for each y in Z, the set $\{x: x < y\}$ is countable. For each y, let $f_y(x)$ be a one-to-one mapping of this set into the positive integers. We can then consider $f(x, y)$ as a mapping from pairs (x, y) of elements of Z such that $x < y$, to integers. Now, for each x in Z and each positive integer, n, let the A_x^n be $\{y: x < y$ and $f(x, y) = n\}$. We can picture these sets as arranged in an infinite matrix with denumerably many rows and uncountably (aleph_1) many columns:

$$\begin{array}{l} A_1^1, A_2^1, \ldots A_n^1 \ldots A_\alpha^1 \ldots \\ A_1^2, A_2^2, \ldots A_n^2 \ldots A_\alpha^2 \ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ A_1^n, A_2^n, \ldots A_n^n, \ldots A_\alpha^n \ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots \end{array}$$

The sets have been constructed so that: (a) The sets in any row are disjoint. (b) The union of the sets in any column is equal to the whole set Z minus a countable set. [(a) follows from the 1-to-1 nature of f considered as a function of x. For (b), any y greater than the x of the column belongs to the set in the column for which $n = f(x, y)$. The union of the sets in the column then differs from Z by the set of elements less than or equal to x, which, by hypothesis, is countable.]

If, in any row there is a non-denumerable number of sets of positive measure then by the correct form of Zeno's (I), Z cannot have finite measure.

If, on the other hand, in every row only a denumerable number of sets have non-zero measure, then only a denumerable number of sets in the whole matrix have non-zero measure since there are only a denumerable number of rows. Then there must be some column which contains all sets of measure zero, since there are a non-denumerable number of columns. The set of elements in Z not contained in the union of the sets in that column must also have measure zero by the correct form of Zeno's (II) since it is the union of a denumerable number of singletons, each of which, by hypothesis, has measure zero.[14]

At this stage of the game, the elimination of non-measurable sets may appear a rather quixotic goal. Lebesgue measure has extended measurability to a far richer domain then Zeno and Aristotle imagined possible. It meshes with an elegant and powerful theory of integration adequate to the needs of the physical sciences. Perhaps Lebesgue measure should be taken as the theory of measure for physical space, and the existence of non-measurable sets should be viewed as just a mildly surprising consequence of the theory rather than as a real difficulty. This is, I believe, the dominant view among mathematicians and mathematical physicists. The real bite of non-measurability comes not in physics or metaphysics, but in epistemology.

Measures of Degree of Belief

Let us turn our attention to probability measures which are meant to represent rational degrees of belief. In this area, questions of measurability take on a new pungency. It is one thing to say that some widely scattered set of points in Euclidean three-dimensional space does not have a natural volume associated with it; another to say that there must be propositions to which there cannot be a degree of belief.

Some of the assumptions used in demonstrating non-measurability also appear in a new light. Translation and congruence invariance appear now not as falsifiable claims about the structure of measure on physical space, but rather as the result of the exercise of someone's epistemological freedom. I wonder which point on a wheel of fortune will be the lowest point when it comes to rest. I come to degrees of belief which are invariant under translation about the circumference. Can it be denied that it is reasonable and proper for me to do so? Again, shouldn't we be able to have rational degrees of belief defined over the subsets of some set of power aleph_1, whether or not $c = \text{aleph}_1$? Furthermore, the Zenonian intuition that only the empty set (here the null proposition) should receive measure zero is supported by a kind of betting argument. Shimony (1955) showed that *regularity* of probability measure is entailed by *strict coherence*: One should reject systems of bets such that one could in no possible circumstance achieve a net gain although one could suffer a net loss.

It is for reasons such as the foregoing that interest in finitely additive, and non-Archimedean measure has largely been generated by the theory of personal probability. De Finetti has consistently rejected countable additivity as a postulate for the theory of personal probability. For a recent spirited defense of finite additivity, see de Finetti (1972 and 1974). Considerations of strict coherence, and of having conditional probabilities well defined in a natural way, can be used to motivate the move to non-Archimedean valued probabilities. For example, see Bernstein and Wattenberg (1969).

Let me back up, and put these questions in their proper setting. Suppose that we have a set, U, whose elements represent mutually exclusive and jointly exhaustive states of affairs. If we think of such states of affairs as individuated in a maximally specific way, we might call the constituents of U 'possible worlds' (but this raises questions which cannot be discussed here). The subsets of this set can be thought of as statements or 'propositions' if propositions are only individuated up to necessary equivalence relative to the original set of possibilities. A set of such 'propositions' closed under negation, conjunction and disjunction is a Boolean algebra of propositions (under countable conjunction and disjunction, and negation, a Boolean sigma algebra). A (finitely additive) measure defined on such a Boolean algebra of propositions which takes values in $[0, 1]$ and which gives a tautology measure 1 and a contradiction measure 0 is a *probability measure*. (I leave open the questions as to whether the algebra need also be a sigma algebra, whether the measure need also be countably additive, and whether $[0, 1]$ is to be taken as a set of standard reals or whether a non-standard model of the reals can also be utilized, these being material to the issues in question.) The question of measurability then is whether the set of all subsets of U can be taken as the appropriate sigma algebra of propositions, or whether we are forced to restrict our probability assignments to some smaller Boolean algebra.[15]

Suppose that degrees of belief are represented (obviously with some idealization) by a numerical-valued function from a Boolean algebra of propositions. There are well-known pragmatic virtues associated with that function being a probability measure. If it is not, and degrees of belief are used in the standard way in determining the fairness of bets, then the agent in question leaves himself open to a Dutch Book: a finite system of bets each of which he considers fair or favorable, such that the net result is a loss no

matter what happens. A belief function which leaves one open to a Dutch Book is said to be incoherent. Coherent belief evaluation functions must be (finitely additive) probability measures. De Finetti argues that there is no comparable coherence argument for countable additivity; and that the imposition of countable additivity as a postulate has the undesirable consequences of (1) creating unmeasurable sets and (2) precluding probability assignments which are perfectly acceptable from a personalistic point of view, e.g. a uniform distribution on a denumerable set of possibilities. Consequently, he develops his theory of personal probability only under the assumption of finite additivity. Savage (1954) does likewise.

The question of the relation of additivity to coherence is not, however, quite so simple. Consider de Finetti's example of the uniform distribution on a denumerable set of possibilities (e.g. what ticket will win in a denumerable lottery). Finite additivity allows the uniform distribution which gives each ticket exactly zero chance of winning, while maintaining probability one that some ticket wins. I would love to have the chance of betting against someone having such a probability assignment. For each ticket, I will bet him \$100 against nothing that it wins; he will consider each of these bets fair. After the lottery, I collect my \$100. If he declines fair bets on the grounds that not betting is just as good, I can do as well offering favorable bets. I will bet \$101 against \$1/2 that the first ticket wins; \$101 against $\$1/2^n$ that the nth ticket wins. After the lottery, I am assured a net winning of at least \$100. The second example reveals clearly what the first may not; that in each case I am assuming sigma-additivity of the *payoff-values* in totaling up my net gain in the infinite system of bets. In fact, if we make these two assumptions: that a denumerable set of bets is permissible and that the payoff-values are sigma-additive, then one can show that the correlative notion of coherence implies countable additivity of the probability measure. The first notice of this fact of which I am aware is in Spielman (1977).

Let a *betting system* be a function from possible states of affairs to payoff values. A *bet* on a proposition p, is a betting system which has a gain, a, associated with every state of affairs in p, and a loss, b, associated with every state of affairs in the negation of p. The *aggregate* of two betting systems, $B_1 \# B_2$ is the betting system which has at each possible state of affairs w, the sum of the payoffs associated with B_1 and B_2:

$$B_1 \# B_2 (w) = B_1 (w) + B_2 (w).$$

Probability is to perform the practical function of placing a value, *expected value*, on bets and betting arrangements when the agent is uncertain as to the state of the world. It would be an attractive property for such evaluations to have that they are *extensional* in the sense that the valuation depend only on the betting system (the function from possible states of affairs to payoff values), and not on how it is described. Otherwise the agent would regard two different prices fair for the same arrangement, and could be systematically exploited by someone who repeatedly bought an arrangement from him cheap and resold it to him dear. Similar considerations support the contention that valuation under uncertainty, *expected value*, should be additive over aggregation:

$$EV(B_1 \# B_2) = EV(B_1) + EV(B_2).$$

For the agent would presumably sell a betting arrangement with expected value of X for X or more, and buy it for X or less. If payoff value is additive, expected value had better be! Now if $p;q$ are mutually exclusive propositions; B_1 and B_2 are bets on p and q respectively at the same stakes, then $B_1 \# B_2$ is a bet on their disjunction $p \vee q$. In particular, let B_1 be the bet that gains a dollar if p, loses nothing otherwise; B_2 be the bet that gains a dollar if q, loses nothing otherwise; B_3 be the bet that gains a dollar if $p \vee q$, loses nothing otherwise. Then $B_1 \# B_2 = B_3$, so by extensionality $EV(B_1 \# B_2) = EV(B_3)$ and by additivity of expected value over aggregation, $EV(B_1) + EV(B_2) = EV(B_3)$. Since these expected values equal by definition the respective probabilities of p, q, and $p \vee q$, we have finite additivity, $\mathrm{pr}(p) + \mathrm{pr}(q) = \mathrm{pr}(p \vee q)$. Now the point of going through all this, is to call attention to the fact that if payoff value is countably additive, then we can consider denumerable aggregates of bets, whose payoffs at each possible state of affairs, w, is the denumerable sum of the payoffs of its constituents:

$$\#_i B_i(w) = \Sigma_i B_i(w),$$

and run the analogous argument for countable additivity.

All the considerations that came into play regarding non-measurable sets in previous sections are now again on the table: finite and countable additivity, invariance, regularity, Archimedean and non-Archimedean values for the measure; domains of various cardinality on which the measure is to be defined. This is not the place to attempt to sort them out. Perhaps enough has been said to show that the truly deep issues first raised by Zeno still deserve to engage our interest.

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NOTES

1 This view of the paradoxes is vigorously advocated by Owen (1957–8).

2 e.g. Grünbaum (1952; 1963; 1968).

3 See Luria (1933), Fränkel (1942), Owen (1957–8), Furley (1967 and 1969), Vlastos (1971).

4 See also Aristotle's long discussion in the *Physics*, Bk. VIII, Ch. 8, 263^a4–264^a6, where Aristotle wrestles with the question of the midpoint and related questions having to do with open and closed intervals, and their relation to change.

5 "The infinite by way of addition is in a manner the same as the infinite by way of division. Within a finite magnitude the infinite by way of addition is realized in an inverse way (to that by way of division); for, as we see the magnitude being divided *ad infinitum*, so, in the same way the sum of successive fractions when added to one another (continually) will be found to tend towards a determinate limit. For if, in a finite magnitude, you take a determinate fraction of it, and then add to that fraction in the same ratio, and so on [i.e., so that each part has to the preceding part the same ratio as the part first taken has to the whole], but *not* each time including (in the part taken) on and the same amount of the original whole, you will not traverse (i.e., exhaust) the finite magnitude. But if you increase the ratio so that it always includes one and the same magnitude, whatever it is, you will traverse it, because any finite magnitude can be exhausted by taking away from it continually any definite magnitude however small" (Aristotle, *Physics*, Bk. III, 6, 206_b, tr. Heath in Heath, 1949, p. 106).

6 Let me put Zeno's statement in context of the fragment which contains it, using Furley's translation: "[Simplicius first summarizes this step in the following words – 'If a thing has no magnitude or bulk (πάχος) or mass, it would not exist.' Then he gives the reasoning in full]. For if it were added to something else that does exist, it would make it no greater; for if it were of no magnitude, and were added, it would not contribute anything to that magnitude. So it would follow that what was added was nothing. If when it is taken away, the other thing is to be no smaller, and is to be no bigger when it is added, it is clear that what was added or taken away was nothing" (Furley, 1967, p. 64). It may help to juxtapose this passage with one from *De generatione et corruptione* where Aristotle is explaining an argument which supposedly led Democritus to a doctrine of indivisible bodies: "Similarly, if it is made out of points, it will not be a quantity. For when they were in contact and there was one magnitude and they were together, they did not increase the magnitude of the whole; for when it was divided into two or more, the whole was no larger or smaller than formerly. So if they are all put together, they will not make a magnitude" (Furley, 1967, p. 84). Furley interprets this passage as arguing "that if a given line is divided in two, the sum of its two parts remains the same as the length of the original whole; yet there are now two points, at the inner end of each of the two half lines, where formerly there was only one; hence the extra point made no difference to the length – and so any number of points will make no difference to the length" (Furley, 1967, p. 85). That is, the argument is that the length of the (closed) line segment $[0, 1/2]$ is exactly $1/2$, as is the magnitude of $[1/2, 1]$. But the magnitude of $[0, 1]$ is exactly 1, so the point $1/2$ which is included in both $[0, 1/2]$ and $[1/2, 1]$ must be exactly zero. Whether or not this is its main purpose, such an argument could certainly be directed at someone who held that the midpoint (and points in general) have infinitesimal magnitude. One cannot help but wonder whether

it is being so used in the pseudo-Aristotelian polemic 'Concerning Indivisible Lines', 970a21: "Further, the addition of the line will not (on the theory) make the whole line any longer than the original line to which the addition was made: For Simples will not, when added together, produce an increased total magnitude" (see also 970b23–25 and 972a12–14).

[7] "A thing that is in succession and touches is 'contiguous.' The 'continuous' is a subdivision of the contiguous: Things are called continuous when the touching limits of each become one and the same and are, as the word implies, contained in each other: continuity is impossible if these extremities are two. This definition makes it plain that continuity belongs to things that naturally in virtue of their mutual contact form a unity. And in whatever way that which holds them together is one, so too will the whole be one, e.g. by a rivet or glue or contact or organic union" (*Physics*, Book V, Ch. 3, 227a6–16).

[8] Open intervals (a, b) and half-open ones $[a, b)$; $(a, b]$ are also assigned measure $b - a$. The endpoint makes things no bigger when added and no smaller when taken away.

[9] Or indeed on any sigma algebra containing the C_rs.

[10] And indeed on all bounded subsets of the reals.

[11] Banach (1923) and Banach and Tarski (1924).

[12] The axes are chosen so that distinct members of the group represent distinct rotations. Hausdorff proves that this is possible.

[13] A, B, C are constructed by recursion on the length of elements in G. 1 is in A; ϕ, ψ in B; ψ^2 in C. Continue as follows:

	x in A	x in B	x in C
x ends in ψ, ψ^2:	$x\phi$ in B	$x\phi$ in A	$x\phi$ in A
x ends in ϕ:	$x\psi$ in B and $x\psi^2$ in C	$x\psi$ in C and $x\psi^2$ in A	$x\psi$ in A and $x\psi^2$ in B

[14] The proof, essentially as I have given it is in Ulam (1930). He then strengthens it by showing that it holds for any set, Z, such that there is no weakly inaccessible cardinal less than or equal in power to Z.

[15] One way to do this would be first to assign a finitely additive probability measure to the sentences of a first-order language, and then extend it to a countably additive probability measure on the sigma algebra generated by the sets of models which satisfy sentences of the language (see Fenstad, 1980).

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SPECIAL RELATIVITY FROM MEASURING RODS

The mathematical structures associated with a space-time theory, such as the special theory of relativity (SRT) – or the general theory (GRT) for that matter – are numerous and interrelated in complex ways.[1] One may start their analysis with the concept of a point set, the elements of which are identified with events in space-time.[2] Imposing a continuity structure on this set leads to the concept of space-time as a four-dimensional topological manifold. Restriction to a differentiable structure then leads to the concept of space-time as a differentiable manifold. Various additional mathematical structures may now be introduced on this manifold: projective, affine, conformal and pseudo-metrical (a metrical structure with Minkowski signature). Each of these mathematical structures is closely associated with the behavior of some idealized physical entity in space-time. The projective structure is associated with the trajectories of structureless free test particles. If each particle carries some intrinsic measure of duration along its trajectory, it reflects the affine structure. The conformal structure is associated with the wave fronts of massless fields, such as the electromagnetic. A pseudo-metrical structure with Minkowski signature implies the existence of two fundamentally distinct types of interval which cannot be transformed into one another by any operation of the symmetry group defining the geometry (the inhomogeneous Lorentz group for SRT).[3] These two distinct types of interval are called spacelike and timelike, and physically quite distinct entities – measuring rods and clocks – are associated with their respective measurement.

While the various mathematical structures introduced in the analysis of space-time are conceptually quite distinct, in both SRT and GRT the projective, affine, conformal and pseudo-metrical structures are inextricably intertwined. One can, for example, mathematically derive all the other structures from the pseudo-metrical structure. Conversely, the pseudo-metrical structure may be derived from compatible projective and conformal structures.[4] In the case of SRT, one may even go a step further and derive the pseudo-metric structure (and thence the projective, of course) from the conformal structure alone, provided certain global assumptions about the entire space-time are added.[5]

This intertwining of projective, affine, conformal and pseudo-metrical

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